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PERTURBATION THEORY FOR LINEAR ELECTROELASTIC EQUATIONS FOR SMALL FIELDS SUPERPOSED ON A BIAS

by

H.F. Tiersten

Office of Naval Research
Contract N00014-76-C-0368
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PERTURBATION THEORY FOR LINEAR ELECTROELASTIC EQUATIONS FOR SMALL FIELDS SUPERPOSED ON A BIAS

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ABSTRACT

A perturbation formulation of the equations of linear piezoelectricity for small fields superposed on a bias is obtained from a Green's function representation. It is shown that the resulting equation for the first perturbation of the eigenvalue may be obtained without the use of a Green's tensor or a complete set of orthogonal eigensolutions. Since the bias enters the constitutive equations, the boundary conditions contain perturbation terms as well as the differential equations. The linear electroelastic equations for small fields superposed on a bias differ from the equations of linear piezoelectricity because the effective material constants of linear electroelasticity have less symmetry than the constants of linear piezoelectricity. Consequently, a perturbation formulation of the linear electroelastic equations for small fields superposed on a bias is presented. It is shown that the effective constants of linear electroelasticity have just the symmetry required for the condition of the orthogonality of linear electroelastic vibrations to hold.

1. Introduction

In vibrating piezoelectric solids small changes in such things as natural frequency and small values of attenuation arising from a variety of causes may readily and conveniently be determined from a perturbation formulation[1-3] of the equations of linear piezoelectricity by employing the equation for the first perturbation of the eigenvalue. This fact is well known and widely appreciated. The small effects may be due to such things as material viscosity, air loading of the surface, mass loading and stiffening of the surface due to the presence of a thin surface film, the electrical conductivity of the film and biasing stresses, strains and electric fields, which either may be applied or a consequence of such things as changes in temperature. Although the linear piezoelectric equations may readily be employed in the treatment of most of the above-mentioned phenomena, they are invalid when the above-mentioned biasing states are present because the linear piezoelectric equations are not equivalent to the linear electroelastic equations for small fields superposed on a bias[4]. The basic reason for this is that although the linear piezoelectric equations are the appropriate linear limit of the properly invariant nonlinear electroelastic equations[5], they are not the appropriate linear limit of the properly invariant nonlinear electroelastic equations for small fields superposed on a bias.

In this paper perturbations from solutions of the linear piezoelectric equations due to the above-mentioned biasing states are treated by means of the aforementioned linear electroelastic equations for small fields superposed on a bias. In the treatment the equations are written as the linear piezoelectric equations plus the additional terms arising from the bias, and a piezoelectric perturbation theory is obtained from a Green's function formulation of the equations of linear piezoelectricity. The derivation is identical with that given in a recent work[2], but the perturbation terms are different and only perturbations due to biasing states are considered. The small effects due to the aforementioned other phenomena are expressly not considered here, but the inclusion of them would be straightforward and may readily be accomplished after the fact. The resulting equation for the first perturbation of the eigenvalue, which is also obtained in a completely independent manner without assuming the existence of a complete set of orthogonal eigensolutions, may readily be applied in the determination of small changes in natural frequencies of vibrating piezoelectric solids subject to various biasing states. Indeed, the perturbation equation has already been applied[6] in the calculation of the change in velocity of piezoelectric surface waves on Y-cut, Z-propagating lithium niobate subject to uniaxial biasing stresses. The results of the perturbation calculation agree so well with complete calculations performed for the same case that it may be concluded that the perturbation procedure is every bit as accurate as the complete calculation for the determination of such quantities. In fact, since the perturbation procedure can readily treat spatially varying biasing states, for which a complete calculation cannot be performed, it has a significant advantage over a complete calculation in the determination of this type of quantity.

Since it is possible for perturbations to arise from the above-mentioned other phenomena even when a biasing state is present and the piezoelectric perturbation theory is invalid under such circumstances, an electroelastic perturbation theory is obtained from a Green's function formulation of the linear electroelastic equations for small fields superposed on a bias. This treatment differs from the previous one in that the terms arising from the bias are included in the basic equations that must be satisfied, rather than as a perturbation from the linear piezoelectric equations. In this linear electroelastic case the effective material constants have less symmetry than in the linear piezoelectric case, but they have just the symmetry required for the orthogonality theorem for linear electroelastic vibrations superposed on a bias, which we need for our perturbation theory, and, of course, the related theorems of reciprocity and uniqueness. The resulting equation for the first perturbation of the eigenvalue may readily be applied in the determination of small changes in natural frequency and phase velocity and small values of attenuation of vibrating biased electroelastic solutions, which arise from what we have termed other phenomena.

2. Linear Electroelastic Equations for Small Fields Superposed on a Bias

Before presenting the equations we briefly introduce some preliminary terminology and notation. We first note that under the static bias the material points move from the reference coordinates X_L to the intermediate coordinates ξ_α, and we have

$$\xi_\alpha = \xi_\alpha(X_L) . \tag{2.1}$$

Then in the superposed small dynamic motion the material points move from the intermediate coordinates ξ_α to the present coordinates y_i, and we have

$$y_i = y_i(\xi_\alpha, t) = \hat{y}_i(X_L, t) . \tag{2.2}$$

We consistently use the convention that capital Latin indices, lower case Greek indices and lower case Latin indices, respectively, refer to the Cartesian components of the reference coordinates, intermediate coordinates and present coordinates of material points. In this paper Cartesian tensor notation is used exclusively. A comma followed by an index denotes partial differentiation with respect to a geometric coordinate, i.e.,

$$\xi_{\alpha,L} = \partial\xi_\alpha/\partial X_L\,, \quad X_{L,\alpha} = \partial X_L/\partial\xi_\alpha\,, \quad y_{i,\alpha} = \partial y_i/\partial\xi_\alpha\,,$$
$$\xi_{\alpha,i} = \partial\xi_\alpha/\partial y_i\,, \quad \hat{y}_{i,L} = \partial\hat{y}_i/\partial X_L\,, \quad X_{L,i} = \partial X_L/\partial\hat{y}_i\,, \tag{2.3}$$

and we employ the summation convention for repeated tensor indices and the dot notation for differentiation with respect to time. Since the dynamic motion is small, we may write

$$y_i = \delta_{i\beta}(\xi_\beta + u_\beta)\,, \tag{2.4}$$

where u_β is the small mechanical displacement from the intermediate coordinate ξ_β to the present coordinate y_i and $\delta_{i\beta}$ is a Kronecker delta, which serves to translate a vector from the present to the intermediate coordinates and vice versa.

By virtue of (2.1) the small field dynamic equations may use either the intermediate coordinates ξ_α or the reference coordinates X_L as independent variables. Since in the typical situation it is undesirable to measure the geometry each time the bias is varied, it is advantageous to use the X_L as independent variables. Moreover, in the applications envisaged although the static biasing mechanical displacement w_L of a material point is large compared with u_β, it is nevertheless small, and we may write

$$\xi_\alpha = \delta_{\alpha L}(X_L + w_L)\,, \tag{2.5}$$

where $\delta_{\alpha L}$ is a Kronecker delta, which serves to translate a vector from the intermediate to the reference coordinates and vice versa and is required for notational consistency and clarity because of the use of different indices to refer to the different coordinates of a material point. In accordance with the foregoing the present electric potential at a material point may be written

$$\varphi(y_j, t) = \hat{\varphi}(X_L, t) = \hat{\varphi}^1(X_L) + \hat{\tilde{\varphi}}(X_L, t), \tag{2.6}$$

where $\hat{\varphi}^1$ is the biasing electric potential at a material point and $\hat{\tilde{\varphi}}$ is the small field dynamic electric potential at the same point.

Since we have used the reference coordinates X_L as the independent variables for material points, we take the X_L rather than the ξ_α as the independent variables for points of free space not abutting a material boundary. Moreover, since only the reference position of the material boundary is known, the value of the electric potential ψ on the free space side of the present position of the material boundary must be obtained by means of a Taylor expansion about its value at the <u>known</u> reference position of the material boundary in the <u>unknown</u> mechanical displacement $(w_M + \delta_{\beta M} u_\beta)$ of the material boundary. Thus

$$\psi(y_j, t) = \hat{\psi}^1(X_M) + \hat{\psi}^1_{,L} w_L + \hat{\psi}^1_{,L} \delta_{\beta L} u_\beta + \tilde{\psi}_{,L} w_L + \tilde{\psi}(X_L, t), \tag{2.7}$$

to first order in the small field variables, and where $\hat{\psi}^1$ is the biasing electric potential immediately on the free space side of the reference position of the material boundary and $\hat{\tilde{\psi}}$ is the small field dynamic electric potential at the same point and for points of free space we have

$$\hat{\psi}(X_L, t) = \psi^1(X_L) + \tilde{\psi}(X_L, t). \tag{2.8}$$

Now that the meaning of the basic dependent and independent variables has been explained we are in a position to write the linear electroelastic equations for small fields superposed on a bias, which take the form[4]

$$\tilde{K}_{L\gamma,L} = \rho^\circ \ddot{u}_\gamma, \quad \tilde{\mathcal{D}}_{L,L} = 0, \tag{2.9}$$

where

$$\tilde{K}_{L\gamma} = \underset{1}{G}_{L\gamma M\alpha} u_{\alpha,M} + \underset{2}{G}_{ML\gamma} \hat{\tilde{\varphi}}_{,M},$$

$$\tilde{\mathcal{D}}_L = \underset{1}{R}_{LM\alpha} u_{\alpha,M} + \underset{2}{R}_{LM} \hat{\tilde{\varphi}}_{,M}, \tag{2.10}$$

and

$$\underset{1}{G}_{L\gamma M\alpha} = \underset{2}{c}_{L\gamma M\alpha} + \underset{2}{\hat{c}}_{L\gamma M\alpha}, \quad \underset{2}{R}_{LM} = -\epsilon_{LM} - \hat{\epsilon}_{LM},$$

$$\underset{2}{G}_{ML\gamma} = \underset{1}{R}_{ML\gamma} = e_{ML\gamma} + \hat{e}_{ML\gamma}, \tag{2.11}$$

$$\underset{2}{\hat{c}}_{L\gamma M\alpha} = T^{1}_{LM}\delta_{\gamma\alpha} + \underset{3}{c}_{L\gamma M\alpha AB} E^{1}_{AB} + \underset{2}{c}_{L\gamma KM} w_{\alpha,K} + \underset{2}{c}_{LKM\alpha} w_{\gamma,K} - \underset{1}{k}_{AL\gamma M\alpha}\hat{\varphi}^{1}_{,A} + \overset{1}{g}_{L\gamma M\alpha},$$

$$\hat{e}_{ML\gamma} = -\underset{1}{k}_{ML\gamma BC} E^{1}_{BC} - e_{MLK} w_{\gamma,K} - b_{AML\gamma}\hat{\varphi}^{1}_{,A} + \overset{2}{g}_{ML\gamma},$$

$$\hat{\epsilon}_{LM} = b_{LMCD} E^{1}_{CD} - \underset{2}{\chi}_{LMC}\hat{\varphi}^{1}_{,C} - 2\epsilon_o J^{1} E^{1}_{ML}, \tag{2.12}$$

and

$$\overset{1}{g}_{L\gamma M\alpha} = J^{1}\epsilon_o[E^{1}_{\zeta}E^{1}_{\zeta}(X_{M,\gamma}X_{L,\alpha} - X_{L,\gamma}X_{M,\alpha}) + E^{1}_{\gamma}(E^{1}_{\beta}X_{L,\beta}X_{M,\alpha} - E^{1}_{\beta}X_{M,\beta}X_{L,\alpha}) + E^{1}_{\alpha}(E^{1}_{\beta}X_{M,\beta}X_{L,\gamma} - E^{1}_{\beta}X_{L,\beta}X_{M,\gamma})],$$

$$\overset{2}{g}_{ML\gamma} = J^{1}\epsilon_o[E^{1}_{\beta}X_{M,\beta}X_{L,\gamma} - E^{1}_{\alpha}X_{L,\alpha}X_{M,\gamma} - E^{1}_{\gamma}X_{M,\alpha}X_{L,\alpha}], \tag{2.13}$$

and we have taken the liberty of changing the order of the last two indices on the second and third rank tensors having both Latin and Greek indices from that employed in Ref.4. Equations (2.9) are the small field stress equations of motion and charge equation of electrostatics referred to the reference coordinates X_L. Equations (2.10) are the linear electroelastic constitutive equations and Eqs. (2.11) and (2.12) contain the definitions of the effective coefficients defined therein. In (2.9) - (2.13) $\tilde{K}_{L\gamma}$ and $\tilde{\mathcal{D}}_L$ denote the components of the small field Piola-Kirchhoff stress tensor, which is asymmetric, and reference electric displacement vector[4], respectively; ρ_o denotes the reference mass density, $\underset{2}{c}_{L\gamma M\alpha}$, $e_{ML\gamma}$ and ϵ_{LM} denote the second order elastic, piezoelectric and dielectric constants, respectively, which are the constants that occur in the ordinary linear theory of piezoelectricity. The symbols T^{1}_{LM} and E^{1}_{AB} denote the components of the static biasing stress and strain, respectively, and

J^1 is the Jacobian of the static deformation. The biasing variables satisfy the appropriate static equations given in (66) - (72) of Ref.4, or the equivalent equations using reference coordinates as independent variables. Since we are interested in small biasing strains only, we have

$$E^1_{AB} = \frac{1}{2}(w_{A,B} + w_{B,A}), \quad J^1 \approx 1, \tag{2.14}$$

$$T^1_{LM} = \underset{2}{c}_{LMRS} w_{R,S} + e_{RLM} \hat{\varphi}^1_{,M}, \tag{2.15}$$

and each $X_{L,\alpha}$ in (2.13) may be replaced by $\delta_{L\alpha}$. In (2.12) $\underset{3}{c}_{L\gamma M\alpha AB}$, $b_{AML\gamma}$, $\underset{2}{\chi}_{LMC}$ and $\underset{1}{k}_{ML\gamma BC}$ denote the third-order elastic, electrostrictive, third-order electric permeability and first-order electroelastic constants, respectively, and ϵ_o denotes the electric permittivity of free space. For obvious reasons the notation employed here is designed to be consistent with the notation of Ref.4. The karets over many variables have been employed here because we consistently use the reference coordinates as independent variables.

To the foregoing equations we must adjoin the dynamic boundary conditions, which when the boundary of the body abuts free space take the form

$$N_L(\tilde{K}_{L\gamma} - \tilde{K}^f_{L\gamma}) = \overline{T}_\gamma, \quad N_L(\tilde{\mathcal{D}}_L - \tilde{\mathcal{D}}^f_L) = 0,$$
$$\tilde{\hat{\varphi}} = \hat{\psi}^1_{,L}\delta_{\beta L}u_\beta + \tilde{\psi}_{,L}w_L + \tilde{\psi}, \tag{2.16}$$

where w_L and u_β in $(2.16)_3$, respectively, denote the static biasing and small field dynamic components of the mechanical displacement at the surface of the solid, N_L denotes the components of the unit normal to the reference position of the surface and $\overline{T}_\gamma$ is an applied traction per unit reference area. The free-space dynamic variables $\tilde{K}^f_{L\gamma}$ and $\tilde{\mathcal{D}}^f_L$ are given by

$$\tilde{K}^f_{L\gamma} = \overset{f}{\underset{1}{G}}_{L\gamma\beta}\, u_\beta + \overset{f}{\underset{2}{G}}_{L\gamma M\alpha}\, u_{\alpha,M} + \overset{f}{\underset{3}{G}}_{L\gamma M}\, \tilde{\psi}_{,M} ,$$

$$\tilde{\mathcal{D}}^f_L = \overset{f}{\underset{1}{R}}_{L\beta} u_\beta + \overset{f}{\underset{2}{R}}_{LM\gamma} u_{\gamma,M} + \overset{f}{\underset{3}{R}}\tilde{\psi}_{,M} , \tag{2.17}$$

where

$$\overset{f}{\underset{1}{G}}_{L\gamma\beta} = \overset{1}{J} X_{L,\alpha}\epsilon_o(\overset{1}{E}_\gamma \overset{1}{E}_{\alpha,\beta} + \overset{1}{E}_{\gamma,\beta}\overset{1}{E}_\alpha - \overset{1}{E}_\delta \overset{1}{E}_{\delta,\beta}\delta_{\alpha\gamma}) ,$$

$$\underset{2}{G}_{L\gamma M\zeta} = \overset{1}{J} X_{L,\alpha} X_{M,\beta}\epsilon_o [-\overset{1}{E}_\beta \overset{1}{E}_\gamma \delta_{\alpha\zeta} + \frac{1}{2}\overset{1}{E}_\kappa \overset{1}{E}_\kappa(\delta_{\gamma\beta}\delta_{\alpha\zeta}$$

$$- \delta_{\zeta\beta}\delta_{\alpha\gamma}) + \overset{1}{E}_\alpha \overset{1}{E}_\gamma \delta_{\zeta\beta}] ,$$

$$\overset{f}{\underset{3}{G}}_{L\gamma M} = \overset{1}{J} X_{L,\alpha} X_{M,\beta}\epsilon_o(\overset{1}{E}_\beta\delta_{\alpha\gamma} - \overset{1}{E}_\gamma\delta_{\alpha\beta} - \overset{1}{E}_\alpha\delta_{\gamma\beta}) ,$$

$$\overset{f}{\underset{2}{R}}_{LM\gamma} = \overset{1}{J} X_{L,\alpha} X_{M,\beta}\epsilon_o(\overset{1}{E}_\alpha\delta_{\gamma\beta} - \overset{1}{E}_\beta\delta_{\gamma\alpha}) ,$$

$$\overset{f}{\underset{1}{R}}_{L\beta} = \overset{1}{J} X_{L,\alpha}\epsilon_o \overset{1}{E}_{\alpha,\beta} , \quad \overset{f}{\underset{3}{R}} = - \overset{1}{J}\epsilon_o . \tag{2.18}$$

In free-space the small field dynamic electric potential clearly satisfies Laplace's equation[4], i.e.,

$$\tilde{\psi}_{,LL} = 0 . \tag{2.19}$$

When the body abuts another solid insulator rather than free-space, instead of (2.16) we must have

$$N_L[\tilde{K}_{L\gamma}] = 0 , \quad N_L[\tilde{\mathcal{D}}_L] = 0 , \quad [u_\gamma] = 0 , \quad [\tilde{\varphi}] = 0 . \tag{2.20}$$

If, on the other hand, the body abuts a perfect conductor, we have

$$N_L[\tilde{K}_{L\gamma}] = 0 , \quad N_L[\tilde{\mathcal{D}}_L] = \tilde{\mathcal{Q}} , \quad \varphi = \bar{\varphi} , \tag{2.21}$$

where $\tilde{\mathcal{D}}_L$ in the conductor vanishes, $\tilde{K}_{L\gamma}$ in the conductor is not given by (2.10), is purely mechanical and depends on the particular case treated and $\tilde{\mathcal{Q}}$ is the dynamic electric surface charge density per unit reference area. In the usual case of interest $\bar{\varphi}$ will be prescribed and $(2.21)_2$ will determine the surface charge density a posteriori.

3. Perturbations from Piezoelectric Solutions

In this section we obtain the equations for perturbations from solutions of the linear piezoelectric equations due to a bias from a Green's function formulation[2] of the equations of linear piezoelectricity. To this end we first write Eqs. (2.9) - (2.12) in the form

$$\tilde{K}^{\ell}_{L\gamma,L} + \tilde{K}^{n}_{L\gamma,L} = \rho^{\circ}\ddot{u}_{\gamma}, \tag{3.1}$$

$$\tilde{\mathscr{D}}^{\ell}_{L,L} + \tilde{\mathscr{D}}^{n}_{L,L} = 0, \tag{3.2}$$

$$\tilde{K}^{\ell}_{L\gamma} = \underset{2}{c}_{L\gamma M\alpha}u_{\alpha,M} + e_{ML\gamma}\tilde{\hat{\varphi}}_{,M},$$

$$\tilde{\mathscr{D}}^{\ell}_{L} = e_{LM\alpha}u_{\alpha,M} - \epsilon_{LM}\tilde{\hat{\varphi}}_{,M}, \tag{3.3}$$

$$\tilde{K}^{n}_{L\gamma} = \hat{c}_{L\gamma M\alpha}u_{\alpha,M} + \hat{e}_{ML\gamma}\tilde{\hat{\varphi}}_{,M},$$

$$\tilde{\mathscr{D}}^{n}_{L} = \hat{e}_{LM\alpha}u_{\alpha,M} - \hat{\epsilon}_{LM}\tilde{\hat{\varphi}}_{,M}, \tag{3.4}$$

where the nonlinear terms $\tilde{K}^{n}_{L\gamma}$ and $\tilde{\mathscr{D}}^{n}_{L}$ are the perturbation terms which depend on the biasing state $w_{A,B}$ and $\hat{\varphi}^{1}_{,M}$ in accordance with (2.12) and $(2.14)_1$. Since $\tilde{K}^{n}_{L\gamma}$ and $\tilde{\mathscr{D}}^{n}_{L}$ are the perturbation terms, we are perturbing from the linear piezoelectric equations and, hence, the equations for the Green's tensor[2] may be written in the form

$$\tilde{K}^{\ell\alpha}_{L\gamma,L} + \rho\omega^{2}G^{\alpha}_{\gamma} = -\,\delta(P-Q)\delta^{\alpha}_{\gamma}, \tag{3.5}$$

$$\tilde{\mathscr{D}}^{\alpha}_{L,L} = -\,\delta(P-Q)\delta^{\alpha}_{4}, \tag{3.6}$$

$$\tilde{K}^{\ell\alpha}_{L\gamma} = \underset{2}{c}_{L\gamma M\zeta}G^{\alpha}_{\zeta,M} + e_{ML\gamma}f^{\alpha}_{,M},$$

$$\tilde{\mathscr{D}}^{\alpha}_{L} = e_{LM\zeta}G^{\alpha}_{\zeta,M} - \epsilon_{LM}f^{\alpha}_{,M}, \tag{3.7}$$

where the superscripts α, $\beta = 1-4$, P and Q denote the fixed field point and variable source point, respectively, δ is the Dirac delta function, δ^{α}_{γ} is the Kronecker delta, G^{α}_{γ} and f^{α} are the mechanical displacement Green's tensor and electric potential Green's function plus cross terms, respectively. In

(3.5) - (3.7) we have assumed that all variables have a time dependence of $e^{i\omega t}$, which has been factored out, and we make the same assumptions in (3.1) - (3.4). Equations (3.1) - (3.3) and (3.5) - (3.7) are identical with Eqs. (2.1) - (2.3) and (2.4) - (2.6) of Ref. 2, except for notation. Consequently, except for notation, Sec. 2 of Ref. 2 applies here without change. Thus, by following a procedure identical with the one employed in Sec. 2 of Ref. 2, from Eq. (2.16) of Ref. 2 with the change in notation we may write

$$u_\alpha = \sum_\mu \frac{g^\mu_\gamma(P)}{(\omega^2_\mu - \omega^2)} \Big[\int_{S_o} N_L [\tilde{K}^\ell_{L\zeta} g^\mu_\zeta(Q) - u_\zeta k^\mu_{L\zeta}(Q) + \tilde{\mathcal{D}}^\ell_L \hat{f}^\mu(Q) - \tilde{\hat{\varphi}} d^\mu_L(Q)] dS_o(Q) + \\ + \int_{V_o} [\tilde{K}^n_{L\zeta,L} g^\mu_\zeta(Q) + \tilde{\mathcal{D}}^n_{L,L} \hat{f}^\mu(Q)] dV_o(Q) \Big], \tag{3.8}$$

where ω_μ is the eigenfrequency of the μth eigensolution and

$$g^\mu_\gamma = u^\mu_\gamma / N_{(\mu)}, \quad \hat{f}^\mu = \tilde{\hat{\varphi}}^\mu / N_{(\mu)}, \tag{3.9}$$

in which u^μ_γ and $\tilde{\hat{\varphi}}^\mu$ denote the μth eigensolution functions satisfying (3.1) and (3.2) in the absence of $\tilde{K}^n_{L\gamma}$ and $\tilde{\mathcal{D}}^n_L$ and subject to the appropriate homogeneous boundary conditions and $N_{(\mu)}$ may be found from[7]

$$\int_{V_o} \rho_o u^\mu_\gamma u^\nu_\gamma \, dV_o = N^2_{(\mu)} \delta_{\mu\nu}, \tag{3.10}$$

which is the orthogonality condition for piezoelectric vibrations. Thus, g^μ_γ and $\hat{f}^\mu$ denote orthonormal eigensolutions of the appropriate homogeneous problem. In (3.8) S_o denotes the reference surface area enclosing the reference volume V_o and

$$k^\mu_{L\gamma} = c_{L\gamma M\varkappa} g^\mu_{\varkappa,M} + e_{ML\gamma} \hat{f}^\mu_{,M},$$
$$d^\mu_L = e_{LM\varkappa} g^\mu_{\varkappa,M} - \epsilon_{LM} \hat{f}^\mu_{,M}. \tag{3.11}$$

We may now obtain the perturbation procedure from (3.8) in the usual way[8], i.e., by letting $u_\gamma(P)$ be very near one of the g^μ_γ, say g^M_γ. Then we may write

$$u_\gamma = g^M_\gamma(P) + \sum_{\mu \neq M} g^\mu_\gamma(P) H_\mu / (\omega^2_\mu - \omega^2) , \tag{3.12}$$

where

$$H_\mu = \int_{S_o} N_L [\tilde{K}^\ell_{L\zeta} g^\mu_\zeta - u_\zeta k^\mu_{L\zeta} + \tilde{\mathscr{D}}^\ell_L \hat{f}^\mu - \tilde{\hat{\varphi}}\, d^\mu_L]\, dS_o + \int_{V_o} [\tilde{K}^n_{L\zeta,L} g^\mu_\zeta + \tilde{\mathscr{D}}^n_{L,L} \hat{f}^\mu]\, dV_o , \tag{3.13}$$

and from (3.8), (3.12) and (3.13) we have

$$H_M / (\omega^2_M - \omega^2) = 1 , \tag{3.14}$$

which is the equation for the first perturbation in eigenfrequency. If

$$\Delta = \omega_M - \omega , \quad |\Delta| \ll \omega_M , \tag{3.15}$$

from (3.14) we have

$$\Delta = H_M / 2\omega_M , \tag{3.16}$$

for the first perturbation in eigenfrequency. We have discussed first-order perturbation theory only because that is all we are interested in here. For a discussion of second-and higher-order perturbation theory see Ref. 8.

The equation for the first perturbation of the eigenvalue, i.e., Eq. (3.16), is a very important relation that has numerous applications. Although it was obtained from a Green's function formulation by assuming the existence of a complete set of orthogonal eigensolutions, this particular relation can readily be obtained without the use of a Green's tensor or, more importantly, a complete set of orthogonal eigensolutions. In fact, the existence of only the particular unperturbed eigensolution under consideration and a nearby perturbed state is required. To see this consider the unperturbed Mth eigensolution, at eigenfrequency ω_M, which, from (3.1) and (3.2), satisfies

$$\tilde{K}^{\ell M}_{L\gamma,L} + \rho^{o}\omega^{2}_{M}u^{M}_{\gamma} = 0, \quad \tilde{\mathcal{D}}^{\ell M}_{L,L} = 0, \tag{3.17}$$

along with the nearby perturbed solution at frequency ω, which, from (3.1) and (3.2), satisfies

$$\tilde{K}^{\ell}_{L\gamma,L} + \tilde{K}^{n}_{L\gamma,L} + \rho^{o}\omega^{2}u_{\gamma} = 0, \quad \tilde{\mathcal{D}}^{\ell}_{L,L} + \tilde{\mathcal{D}}^{n}_{L,L} = 0, \tag{3.18}$$

where it is understood that (3.17) and (3.18) are independent of time. From $(3.17)_1$ and $(3.18)_1$ form the equation

$$\int_{V_o} [(\tilde{K}^{\ell M}_{L\gamma,L} + \rho^{o}\omega^{2}_{M}u^{M}_{\gamma})u_{\gamma} - (\tilde{K}^{\ell}_{L\gamma,L} + \tilde{K}^{n}_{L\gamma,L} + \rho^{o}\omega^{2}u_{\gamma})u^{M}_{\gamma}]\, dV_o = 0. \tag{3.19}$$

Performing the usual operations[7], employing (3.3), $(3.17)_2$ and $(3.18)_2$ and the divergence theorem in the usual manner, we obtain

$$(\omega^{2}_{M} - \omega^{2}) \int_{V_o} \rho^{o}u^{M}_{\gamma}u_{\gamma}\, dV_o = \int_{S_o} N_L[\tilde{K}^{\ell}_{L\gamma}u^{M}_{\gamma} - \tilde{K}^{\ell M}_{L\gamma}u_{\gamma} + \tilde{\mathcal{D}}^{\ell}_{L}\tilde{\varphi}^{M} - \tilde{\mathcal{D}}^{\ell M}_{L}\tilde{\varphi}]\, dS_o$$
$$+ \int_{V_o} [\tilde{K}^{n}_{L\gamma,L}u^{M}_{\gamma} + \tilde{\mathcal{D}}^{n}_{L,L}\tilde{\varphi}^{M}]\, dS_o. \tag{3.20}$$

Since the perturbed solution is nearby the unperturbed solution, we have

$$\Delta = \omega_M - \omega, \quad |\Delta| \ll \omega_M, \quad u^{M}_{\gamma} - u_{\gamma} = \eta_{\gamma}, \quad |\eta_{\gamma}| \ll |u^{M}_{\gamma}|. \tag{3.21}$$

Substituting from (3.21) into (3.20), neglecting products of small quantities and employing (3.9) - (3.11) and (3.13), we obtain

$$\Delta = H_M/2\omega_M, \tag{3.22}$$

which is identical with Eq. (3.16). However, it should be emphasized that although Eq. (3.16) does not require the existence of a complete set of orthogonal eigensolutions, Eq. (3.12) does require the existence of such a set. One consequence of this is that second and higher order perturbation theory[8] cannot be obtained from the procedure presented in this paragraph.

From Eq. (3.16) [or (3.22)] for the first perturbation in eigenfrequency, we can obtain equations for the first perturbation in phase velocity and wavenumber by following the procedure outlined near the end of Sec. 2 of Ref. 2. The equations are

$$\epsilon = \Delta/\xi_M , \quad \delta = \Delta/V_M , \tag{3.23}$$

where

$$V = V_M - \epsilon , \quad \xi = \xi_M + \delta , \tag{3.24}$$

and V_M and ξ_M are the unperturbed phase velocity and wavenumber, respectively, of the Mth eigensolution.

As an example of the application of (3.13), (3.15) and (3.16) [or (3.22)] to a specific case consider a piezoelectric solid with traction free surfaces and for simplicity a sufficiently high dielectric tensor that the normal component of electric displacement can be taken to vanish. Let the solid be subject to static biasing stresses, strains and/or electric fields applied in such a way as not to affect the homogeneous mechanical or electrical boundary conditions determining the normal modes of vibration of the solid. Since the Mth piezoelectric eigensolution perturbed by the bias is for traction free boundary conditions and vanishing normal component of electric displacement, we have

$$N_L k^M_{L\gamma} = 0 , \quad N_L d^M_L = 0 , \tag{3.25}$$

and under these circumstances, from (3.13), we find that H_M takes the form

$$H_M = \int_{S_o} N_L [\tilde{K}^{\ell}_{L\zeta} g^M_\zeta + \tilde{\mathscr{D}}^{\ell}_L \hat{f}^M] \, dS_o + \int_{V_o} [\tilde{K}^n_{L\zeta,L} g^M_\zeta + \tilde{\mathscr{D}}^n_{L,L} \hat{f}^M] \, dV_o . \tag{3.26}$$

The quantities $N_L \tilde{K}^{\ell}_{L\zeta}$ and $N_L \tilde{\mathscr{D}}^{\ell}_L$ in the surface integral in (3.26) are surface perturbation terms, which are to be determined from the Mth eigensolution due to the presence of the bias as are the terms in the volume integral in (3.26).

For traction free and zero normal component of electric displacement boundary conditions in the presence of the bias, we have

$$N_L(\tilde{K}^{\ell}_{L\zeta} + \tilde{K}^{n}_{L\zeta}) = 0, \quad N_L(\tilde{\mathscr{D}}^{\ell}_L + \tilde{\mathscr{D}}^{n}_L) = 0, \tag{3.27}$$

which with (3.26) and the divergence theorem yields

$$H_M = -\int_{V_o} [\tilde{K}^n_{L\zeta} g^M_{\zeta,L} + \tilde{\mathscr{D}}^n_L \hat{f}^M_{,L}] \, dV_o, \tag{3.28}$$

which is the form taken by the perturbation integral in this special but important case. In (3.28) $\tilde{K}^n_{L\zeta}$ and $\tilde{\mathscr{D}}^n_L$ take the values given by the Mth orthonormal piezoelectric eigensolution g^M_ζ, $\hat{f}^M$ in the presence of the bias and, consequently, from (3.4) we have

$$\tilde{K}^n_{L\zeta} = \hat{c}_{L\zeta R\gamma} g^M_{\gamma,R} + \hat{e}_{RL\zeta} \hat{f}^M_{,R},$$
$$\tilde{\mathscr{D}}^n_L = \hat{e}_{LR\gamma} g^M_{\gamma,R} - \hat{\varepsilon}_{LR} \hat{f}^M_{,R}, \tag{3.29}$$

the substitution of which in (3.28) yields

$$H_M = -\int_{V_o} [\hat{c}_{L\zeta R\gamma} g^M_{\gamma,R} g^M_{\zeta,L} + 2\hat{e}_{RL\zeta} \hat{f}^M_{,R} g^M_{\zeta,L} - \hat{\varepsilon}_{LR} \hat{f}^M_{,L} \hat{f}^M_{,R}] \, dV_o. \tag{3.30}$$

Thus, if the piezoelectric eigensolution and bias are known, the perturbed frequency ω can be determined from (3.15), (3.16) and (3.30).

4. Perturbations from Electroelastic Solutions

In this section we obtain the equations for perturbations from solutions of the linear electroelastic equations for small fields superposed on a bias from a Green's function formulation of the equations of linear electroelasticity. To this end we first write Eqs. (2.9) and (2.10) in the form

$$\tilde{K}_{L\gamma,L} + \tilde{K}^{e}_{L\gamma,L} = \rho^{o}\ddot{u}_{\gamma}, \tag{4.1}$$

$$\tilde{\mathcal{D}}_{L,L} + \tilde{\mathcal{D}}^{e}_{L,L} = 0, \tag{4.2}$$

$$\tilde{K}_{L\gamma} = \underset{1}{G}_{L\gamma M\zeta}u_{\zeta,M} + \underset{2}{G}_{ML\gamma}\tilde{\tilde{\varphi}}_{,M},$$

$$\tilde{\mathcal{D}}_{L} = \underset{2}{G}_{LM\zeta}u_{\zeta,M} + \underset{2}{R}_{LM}\tilde{\tilde{\varphi}}_{,M}, \tag{4.3}$$

and note that the definitions in (2.11) - (2.13) hold. In (4.1) and (4.2) we have taken the liberty of introducing the extra quantities $\tilde{K}^{e}_{L\gamma}$ and $\tilde{\mathcal{D}}^{e}_{L}$, which denote mechanical and electrical perturbation terms, respectively, and can, in particular, represent the influence of small material viscosity. From $(2.11)_1$, $(2.12)_{1,3}$ and (2.13) we note that

$$\underset{1}{G}_{L\gamma M\zeta} = \underset{1}{G}_{M\zeta L\gamma}, \quad \underset{2}{R}_{LM} = \underset{2}{R}_{ML}. \tag{4.4}$$

Before presenting the Green's function formulation of the linear electroelastic equations for small fields superposed on a bias, we show that the vibrational eigensolutions satisfy an orthogonality condition because we need this result to obtain the full perturbation theory. To this end consider two eigensolutions of (4.1) and (4.2) minus the perturbation terms containing $\tilde{K}^{e}_{L\gamma}$ and $\tilde{\mathcal{D}}^{e}_{L}$, one solution with eigenfrequency ω_{μ} and the other with ω_{ν}. They satisfy the respective equations

$$\tilde{K}^{\mu}_{L\gamma,L} + \rho^{o}\omega_{\mu}^{2}u^{\mu}_{\gamma} = 0, \quad \tilde{\mathcal{D}}^{\mu}_{L,L} = 0, \tag{4.5}$$

$$\tilde{K}^{\nu}_{L\gamma,L} + \rho^{o}\omega_{\nu}^{2}u^{\nu}_{\gamma} = 0, \quad \tilde{\mathcal{D}}^{\nu}_{L,L} = 0. \tag{4.6}$$

From $(4.5)_1$ and $(4.6)_1$ form the equation

$$\int_{V_o} [(\tilde{K}^{\mu}_{L\gamma,L} + \rho^{o}\omega_{\mu}^{2}u^{\mu}_{\gamma})u^{\nu}_{\gamma} - (\tilde{K}^{\nu}_{L\gamma,L} + \rho^{o}\omega_{\nu}^{2}u^{\nu}_{\gamma})u^{\mu}_{\gamma}]\, dV_o = 0. \tag{4.7}$$

Performing the usual operations[7], employing (4.3), $(4.5)_2$ and $(4.6)_2$ and the divergence theorem in the usual manner, we obtain

$$\int_{S_o} N_L [\tilde{K}^{\mu}_{L\gamma} u^{\nu}_{\gamma} - \tilde{K}^{\nu}_{L\gamma} u^{\mu}_{\gamma} + \tilde{\mathcal{D}}^{\mu}_{L} \tilde{\varphi}^{\nu} - \tilde{\mathcal{D}}^{\nu}_{L} \tilde{\varphi}^{\mu}] \, dS_o = (\omega^2_{\nu} - \omega^2_{\mu}) \int_{V_o} \rho^o u^{\mu}_{\gamma} u^{\nu}_{\gamma} \, dV_o . \tag{4.8}$$

It should be noted that (4.8) can readily be obtained even if intermediate surfaces of discontinuity exist. Clearly, from (4.8), for homogeneous boundary conditions, we have

$$\int_{V_o} \rho^o u^{\mu}_{\gamma} u^{\nu}_{\gamma} \, dV_o = N^2_{(\mu)} \delta_{\mu\nu}, \tag{4.9}$$

which is the orthogonality condition for linear electroelastic vibrations superposed on a bias.

Since $\tilde{K}^e_{L\gamma}$ and $\tilde{\mathcal{D}}^e_L$ are perturbation terms, the equations for the Green's tensor[9] for the linear electroelastic equations for small fields superposed on a bias may be written in the form

$$\tilde{K}^{\alpha}_{L\gamma,L} + \rho^o \omega^2 G^{\alpha}_{\gamma} = - \delta(P - Q) \delta^{\alpha}_{\gamma} , \tag{4.10}$$

$$\tilde{\mathcal{D}}^{\alpha}_{L,L} = - \delta(P - Q) \delta^{\alpha}_{4} , \tag{4.11}$$

$$\tilde{K}^{\alpha}_{L\gamma} = {}_1G_{L\gamma M\zeta} G^{\alpha}_{\zeta,M} + {}_2G_{ML\gamma} f^{\alpha}_{,M} ,$$

$$\tilde{\mathcal{D}}^{\alpha}_{L} = {}_2G_{LM\zeta} G^{\alpha}_{\zeta,M} + {}_2R_{LM} f^{\alpha}_{,M}, \tag{4.12}$$

where equivalent quantities are defined as in the wording following Eq. (3.7). In (4.10) - (4.12) we have assumed that all variables have a time dependence $e^{i\omega t}$. We now make the same assumptions in (4.1) - (4.3) and in the usual manner, from (4.1) and (4.10) we form

$$\int_{V_o} [(\tilde{K}_{L\gamma,L} + \tilde{K}^e_{L\gamma,L} + \rho^o \omega^2 u_{\gamma}) G^{\varkappa}_{\gamma} - (\tilde{K}^{\varkappa}_{L\gamma,L} + \rho^o \omega^2 G^{\varkappa}_{\gamma} + \delta(P - Q) \delta^{\varkappa}_{\gamma}) u_{\gamma}] dV_o(Q) , \tag{4.13}$$

where $\varkappa = 1-3$ and all variables in (4.13) have spatial dependence only. Performing the usual operations[7], employing (4.2), (4.3), (4.11) and (4.12) and the divergence theorem in the usual manner, we obtain

$$u_\varkappa(P) = \int_{S_o} N_L[\tilde{K}_{L\gamma}G^\varkappa_\gamma - \tilde{K}^\varkappa_{L\gamma}u_\gamma + \tilde{\mathcal{D}}_L f^\varkappa - \tilde{\mathcal{D}}^\varkappa_L\tilde{\hat{\varphi}}]\, dS_o(Q) + \int_{V_o} [\tilde{K}^e_{L\gamma,L}G^\varkappa_\gamma + \tilde{\mathcal{D}}^e_{L,L}f^\varkappa]\, dV_o(Q) \,. \tag{4.14}$$

Similarly, from (4.2) and (4.11) we form

$$\int_{V_o} [(\tilde{\mathcal{D}}_{L,L} + \tilde{\mathcal{D}}^e_{L,L})f^4 - (\tilde{\mathcal{D}}^4_{L,L} + \delta(P-Q))\tilde{\hat{\varphi}}]\, dV_o(Q) \,, \tag{4.15}$$

and performing the usual operations[7], utilizing the divergence theorem and employing (4.1), (4.3), (4.10) and (4.12), we obtain

$$\tilde{\hat{\varphi}}(P) = \int_{S_o} N_L[\tilde{\mathcal{D}}_L f^1 - \tilde{\mathcal{D}}^4_L\tilde{\hat{\varphi}} + \tilde{K}_{L\gamma}G^4_\gamma - \tilde{K}^4_{L\gamma}u_\gamma]\, dS_o(Q) + \int_{V_o} [\tilde{\mathcal{D}}^e_{L,L}f^4 + \tilde{K}^e_{L\gamma,L}G^4_\gamma]\, dV_o(Q) \,. \tag{4.16}$$

Equations (4.14) and (4.16) constitute the Green's function (or tensor) formulation of the linear electroelastic equations for small fields superposed on a bias. It turns out that for our purposes, although we envisage use for (4.14) we do not envisage use for (4.16) because of the particular type of perturbation problem in which we are interested.

We now assume that a complete set of eigensolutions u^μ_γ, $\tilde{\hat{\varphi}}^\mu$ exists and define orthonormal eigensolutions to our eigenvibration problem by

$$\hat{g}^\mu_\gamma = u^\mu_\gamma/N_{(\mu)} \,, \quad \tilde{\hat{f}}^\mu = \tilde{\hat{\varphi}}^\mu/N_{(\mu)} \,. \tag{4.17}$$

We now expand the mechanical displacement Green's tensor $G^\varkappa_\gamma$ and electric potential Green's vector $f^\varkappa$ in the forms

$$G_\gamma^\varkappa = \sum_\mu M_\mu^\varkappa \hat{g}_\gamma^\mu, \quad f^\varkappa = \sum_\mu M_\mu^\varkappa \tilde{\hat{f}}^\mu, \tag{4.18}$$

where $\hat{g}_\gamma^\mu$ and $\tilde{\hat{f}}^\mu$ constitute orthonormal solution functions satisfying the appropriate homogeneous form of (4.1) - (4.3) subject to the appropriate homogeneous boundary conditions. Substituting from (4.18) into (4.10), employing (4.12) and the homogeneous form of (4.10) for every μ, contracting with $\hat{g}_\gamma^\mu$, integrating over V_o and utilizing (4.9), we obtain

$$M_\mu^\varkappa = \hat{g}_\varkappa^\mu(P)/(\omega_\mu^2 - \omega^2)\,. \tag{4.19}$$

Substituting from (4.18) and (4.19) into (4.14), we obtain

$$u_\varkappa = \sum_\mu \hat{g}_\varkappa^\mu(P)\hat{H}_\mu/(\omega_\mu^2 - \omega^2)\,, \tag{4.20}$$

where

$$\hat{H}_\mu = \int_{S_o} N_L[\tilde{K}_{L\gamma}\hat{g}_\gamma^\mu(Q) - u_\gamma \hat{k}_{L\gamma}^\mu(Q) + \tilde{\mathcal{D}}_L \tilde{\hat{f}}^\mu(Q) - \tilde{\varphi}\hat{d}_L^\mu(Q)]\, dS_o(Q)$$
$$+ \int_{V_o} [\tilde{K}^e_{L\gamma,L}\hat{g}_\gamma^\mu(Q) + \tilde{\mathcal{D}}^e_{L,L}\tilde{\hat{f}}^\mu(Q)]\, dV_o(Q)\,, \tag{4.21}$$

and

$$\hat{k}_{L\gamma}^\mu = \underset{1}{G}_{L\gamma M\zeta}\hat{g}^\mu_{\zeta,M} + \underset{2}{G}_{ML\gamma}\tilde{\hat{f}}^\mu_{,M}\,,$$
$$\hat{d}_L^\mu = \underset{2}{G}_{LM\zeta}\hat{g}^\mu_{\zeta,M} + \underset{2}{R}_{LM}\tilde{\hat{f}}^\mu_{,M}\,. \tag{4.22}$$

The perturbation procedure is obtained from (4.20) by following the procedure outlined in Sec.3, after Eq.(3.11), and the resulting equation for the first perturbation of the eigenvalue takes the form

$$\Delta = \hat{H}_M/2\omega_M\,. \tag{4.23}$$

Again, as in Sec.3 and by following a procedure directly analogous to the one employed in obtaining (3.22), it may readily be shown that (4.23) can be obtained without the use of a Green's tensor or a complete set of orthogonal eigensolutions.

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